Random Walks on High Dimensional Expanders I

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Input: Undirected graph G

Output: Uniformly random spanning tree of G

Confluence of linear algebra, combinatorics, probability, …

Connections to random walks, effective resistance, statistical physics, …

Algorithms for sampling spanning trees

Exact sampling:

Kirchhoff (matrix tree theorem) [Kir47]

Aldous [Ald90], Broder [Bro89], Wilson [Wil96], ...,

Schild [Sch18] (nearly linear time)

Approximate sampling:

Anari–Liu–Oveis Gharan–Vinzant [ALOGV18], Cryan–Guo–Mousa [CGM19]

Bases-exchange walk

Let T_0 be an arbitrary spanning tree of G

For $t = 0, 1, 2, \ldots$

Remove an edge from T_t uniformly at random to obtain F_t

Among all spanning trees containing F_t, uniformly pick one as new T_{t+1}

Corresponds to a random walk on an auxiliary graph

Mihail–Vazirani conjecture [MV89] (implies) the above walk gives an approximately uniformly random spanning tree in polynomial time

Simplicial complex

Bases-exchange walk is now considered as a random walk on a simplicial complex

Definition (Abstract simplicial complex)

A downward closed family Y of sets over a universe

Downward closed means $A \in Y$ and $B \subseteq A$ imply $B \in Y$

A family of sets is also called a hypergraph

"High dimensional" generalization of graphs to capture non-binary relations

Given a graph G, $Y = \{F \subseteq E(G) \mid F \text{ is acyclic}\}$ is a simplicial complex

Faces of a simplicial complex

Decompose $Y = X(-1) \cup \cdots \cup X(d)$, where

$$X(k) = \{A \in Y \mid |A| = k+1\} \qquad \text{(faces of dimension } k\text{)}$$

It's more natural to consider $Y(k) = X(k-1)$ (subsets of size k)

1-dimensional simplicial complex $\equiv$ graph

We will consider only d-dimensional simplicial complex that is pure,
i.e. every face of Y is a subset of some face of the dimension d

Spectral gap and mixing time

Let P denote the transition matrix of the random walk on a graph, and π its stationary distribution (always exists)

Mixing time t_{mix} = minimum time t such that, for any initial state x,

$$\|P^t(x, \cdot) - \pi\|_1 \leqslant 1/4$$

Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of P

Largest eigenvalue $\lambda_1 = 1$ always, with left eigenvector π

Let $\beta = 1 - \max\{|\lambda_2|, |\lambda_n|\}$ be the (two-sided) spectral gap

It is well known that

$$t_{\text{mix}} \leqslant \frac{1}{\beta} \log\left(\frac{4}{\pi_*}\right)$$

where $\pi_* = \min_x \pi(x)$

Let $Y(k) = \{F \subseteq E(G) \mid |F| = k, F \text{ is acyclic}\}$

Transition P_{BE} of bases-exchange walk consists of two sub-transitions:

1. Down D_{n-1}: From $T_t \in Y(n-1)$ to $F_t \in Y(n-2)$ by dropping a random element (edge)
2. Up U_{n-1}: From F_t to $T_{t+1} \in Y(n-1)$ by choosing $T_{t+1} \supset F_t$

Fact

AB and BA have the same nonzero spectrum for any matrices A and B

Bounding spectral gap of $P_{\text{BE}} = D_{n-1} U_{n-1}$ is the same as bounding spectral gap of $U_{n-1} D_{n-1}$

Up and down transitions for general k

Weight $w(F)$ of any face/forest will be defined later

Down transition D_k from $F \in Y(k)$ to $Y(k-1)$: drop a uniformly random element from F

Up transition U_k from $F' \in Y(k-1)$ to $F \in Y(k)$: go to a random neighbor $F \supset F'$ with probability proportional to $w(F)$

Theorem (Kaufman–Oppenheim [KO18])

$D_k U_k$ has (one-sided) spectral gap at least $1/k$ for $1 \leqslant k \leqslant n-1$

This theorem implies fast mixing of (lazy version of) bases-exchange walk [ALOGV18]

We will see key steps of its proof later

Sampling matroid bases

Definition (Matroid)

A nonempty family M of sets over a universe such that M is downward closed and "exchangable":

If $A, B \in M$ and $|A| > |B|$, then for some $e \in A \setminus B$, we have $B \cup \{e\} \in M$

An inclusion-maximal set in a matroid M is a basis

All bases have the same size, called the rank of M

A matroid is also a pure simplicial complex

Kaufman–Oppenheim lowerbound of $1/k$ also holds for any matroid of rank k

Hence bases-exchange walk for that any matroid mixes rapidly

Aside: Log-concavity

Conjecture (Mason [Mas72])

For any matroid with I_k sets of size k,

$$I_k^2 \geqslant I_{k-1} I_{k+1}$$

First proved by Adiprasito, Huh, Katz [AHK15]

A stronger form later proved by Huh, Schröter, Wang [HSW18]

An even stronger form proved independently by Anari, Liu, Oveis Gharan, Vinzant [ALOGV18'] and Brändén and Huh [BH18]

These works use some related concepts that we will also see

Interesting read: *A Path Less Taken to the Peak of the Math World*, Quanta Magazine