Random Walks on High Dimensional Expanders II

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Recap

A simplicial complex Y is a downward closed family of sets

e.g. $Y = \{F \subseteq E(G) \mid F \text{ is acyclic}\}$

Level $Y(k) = \{F \in Y \mid |F| = k\}$

Down transition D_k from $Y(k)$ to $Y(k-1)$ by dropping a random element

Up transition U_k from $F' \in Y(k-1)$ to $F \in Y(k)$ by choosing $F \supset F'$ with probability proportional to $w(F)$

Theorem (Kaufman–Oppenheim [KO18])

$U_k D_k$ has (one-sided) spectral gap at least $1/k$ for $1 \leqslant k \leqslant n-1$

Skeleton

Given a simplicial complex Y, the 1-skeleton of Y is the graph with vertex set $Y(1)$ and edge set $Y(2)$

Weighted simplicial complex

A distribution π on top layer $Y(d)$ induces a distribution π_k on $Y(k)$ for any $k \leqslant d$:

1. Draw $F \in Y(d)$ from π
2. Uniformly discard all but k elements from F

Generalizes the typical definition for graphs ($\equiv 1$-dimensional simplicial complex)

A graph with weight π (i.e. total edge weight 1) has stationary distribution π_1

Random walk on simplicial complex

Theorem (Kaufman–Oppenheim [KO18])

$U_k D_k$ has (one-sided) spectral gap at least $1/k$ for $1 \leqslant k \leqslant n-1$

Random walk on simplicial complex usually refers to transitions between layers such as $U_k D_k$

In particular, when $k=2$, $U_2 D_2$ $(Y(1) \mapsto Y(2) \mapsto Y(1))$ coincides with the lazy random walk transition on the 1-skeleton

Definition (Lazy random walk)

Given a random walk with transition P, its lazy version is:

- With probability $1/2$, stay at the current vertex
- With probability $1/2$, move according to P

$U_k D_k$ is “high dimensional” lazy random walk

Inner product for functions on graph

Commonly appear in analysis of random walk

Given weighted graph G, define inner product $\langle \cdot, \cdot \rangle_G$ on $V(G) \to \mathbb{R}$ by

$$\langle f, g \rangle_G = \mathop{\mathbb{E}}_{x \sim \pi_1} [f(x) g(x)]$$

Random walk transition matrix P yields the averaging operator

$$Pf(x) = \mathop{\mathbb{E}}_{y \sim x} [f(y)]$$

P is self-adjoint wrt $\langle \cdot, \cdot \rangle_G$:

$$\langle f, Pg \rangle_G = \langle Pf, g \rangle_G \qquad \left(= \mathop{\mathbb{E}}_{x \sim y} [f(x) g(y)] \right)$$

As a self-adjoint operator, P has $|V(G)|$ real eigenvalues and orthogonal eigenvectors (spectral theorem)

Inner product for functions on simplicial complex

Extend the previous definition by considering the 1-skeleton of Y

Given weighted simplicial complex, define inner product $\langle \cdot, \cdot \rangle_Y$ on $Y(1) \to \mathbb{R}$

$$\langle f, g \rangle_Y = \mathop{\mathbb{E}}_{x \sim \pi_1} [f(x) g(x)]$$

Random walk transition matrix P on the 1-skeleton yields

$$Pf(x) = \mathop{\mathbb{E}}_{(x,y) \sim \pi_2} [f(y)]$$

P is self-adjoint wrt $\langle \cdot, \cdot \rangle_Y$:

$$\langle f, Pg \rangle_Y = \langle Pf, g \rangle_Y \qquad \left(= \mathop{\mathbb{E}}_{(x,y) \sim \pi_2} [f(x) g(y)] \right)$$

As a self-adjoint operator, P has $|Y(1)|$ real eigenvalues and orthogonal eigenvectors (spectral theorem)

Link

Given face F of simplicial complex Y, the link of F is the simplicial complex

$$Y_F = \{H \setminus F \mid H \in Y, H \supseteq F\}$$

If Y has weight π, then Y_F has weight π_F, where

$$\pi_F(H \setminus F) = \frac{\pi(H)}{\pi(F)}$$

Recall that π_k is the random process of picking $H \sim \pi$ and dropping elements

$\pi_F(H \setminus F)$ measures the conditional probability of getting F at the end, after discarding $H \setminus F$

When Y is a matroid, Y_F is known as a contraction

Oppenheim's descent theorem

We will only consider one-sided spectral gap

Theorem (Oppenheim [Opp18])

Suppose pure simplicial complex Y with top layer $Y(3)$ satisfies

- *1-skeleton of Y is connected*
- *Every link Y_z for $z \in Y(1)$ has spectral gap at least β*

Then 1-skeleton of Y has spectral gap at least $\dfrac{\beta}{1-\beta}$

The above theorem is used (with extra arguments) to prove

Theorem (Kaufman–Oppenheim [KO18])

$U_k D_k$ has spectral gap at least $1/k$ for $1 \leqslant k \leqslant n-1$

Let G be a weighted graph on n vertices

Its random walk transition matrix P has n real eigenvalues and orthogonal eigenvectors wrt $\langle \cdot, \cdot \rangle_G$

Largest eigenvalue $\lambda_1 = 1$ always, with right eigenvector $\mathbb{1}$

Second eigenvalue $\lambda_2 \leqslant \lambda$ is equivalent to $\langle f, Pf \rangle \leqslant \lambda \langle f, f \rangle$ whenever $f \perp \mathbb{1}$

Descent of spectral gap

$$\langle f, g\rangle_Y = \mathop{\mathbb{E}}_{x\sim\pi_1}[f(x)g(x)] = \mathop{\mathbb{E}}_{(x,y)\sim\pi_2}[f(x)g(x)] = \mathop{\mathbb{E}}_{y\sim\pi_1}\mathop{\mathbb{E}}_{x\sim Y_y(1)}[f(x)g(x)]$$
$$= \mathop{\mathbb{E}}_{y\sim\pi_1}\langle f, g\rangle_{Y_y}$$

Equate inner product over Y to (average of) inner products over links

$$\langle Pf, g\rangle_Y = \mathop{\mathbb{E}}_{(x,y)\sim\pi_2}[f(x)g(y)] = \mathop{\mathbb{E}}_{(x,y,z)\sim\pi_3}[f(u)g(z)] = \mathop{\mathbb{E}}_{z\sim\pi_1}\mathop{\mathbb{E}}_{(x,y)\sim Y_z(2)}[f(x)g(y)]$$
$$= \mathop{\mathbb{E}}_{z\sim\pi_1}\langle P_z f, g\rangle_{Y_z}$$

Spectral gap and variance decay

Let π be stationary distribution of graph G (not edge distribution anymore)

Define variance

$$\mathsf{Var}_\pi(f) = \mathbb{E}_\pi[f^2] - \mathbb{E}_\pi[f]^2$$

Consider distribution $\mu = f\pi$ with density f (after some transitions of the random walk)

Larger spectral gap implies stronger bound on the decay of variance (and hence mixing time)

$$t_{\mathsf{mix}} \leqslant \frac{1}{\beta} \log\left(\frac{4}{\pi_*}\right)$$

Modified log-Sobolev constant

Define (relative) entropy

$$\mathsf{Ent}_\pi(f) = \mathbb{E}_\pi[f \log f] - \mathbb{E}_\pi[f] \log \mathbb{E}_\pi[f]$$

Given distribution $\mu = f\pi$ with density f

$$D(\mu \| \pi) = \mathsf{Ent}_\pi(f)$$

Bobkov and Tetali defined modified log-Sobolev constant ρ to bound the decay of entropy

Theorem (Bobkov–Tetali [BT06])

$$t_{\mathsf{mix}} \leqslant \frac{1}{\rho}\left(\log\log\frac{1}{\pi_*} + \log 32\right)$$

Modified log-Sobolev constant for bases-exchange walk

Theorem (Cryan–Guo–Mousa [CGM19])
The bases-exchange walk on a matroid of rank k has modified log-Sobolev constant at least $1/k$

This resulting mixing time bound is sharp for some matroid

Therefore sampling of spanning tree mixes in $O(|V| \log |V|)$ iterations

Building on this, Anari–Liu–Oveis Gharan–Vinzant [ALOGV20] gave another bases-exchange algorithm to sample spanning tree in $O(|E| \log^2 |E|)$ time