Random Walks on High Dimensional Expanders III

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High dimensional expanders (HDX)

expander graphs:

Eigenvalue gap of transition matrix/normalized Laplacian, edge expansion ...

High dimensional expanders:

Simplicial complex whose underlying graph (1-skeleton) of every link has eigenvalue gap [Dinur–Kaufman17, ...], ... (many other definitions)

Theorem ([Anari–Liu–Oveis Gharan–Vinzant18] + [Kaufman–Oppenheim18])

Matroids are high dimensional expanders

Complete simplicial complex with top level of size d?

Sparse random simplicial complexes?

$O_d(n)$-sized by Dinur–Kaufman [DK17], Based on Ramanujan complexes

Kaufman–Oppenheim [KO18]: elementary algebraic construction

Chapman–Linial–Peled [CLP18], Liu–Mohanty–Yang [LMY19]: combinatorial constructions

Agreement Testing / Direct Product Test

Crucial to locally testable codes and probabilistically checkable proofs

Fix code $C \subseteq \{0,1\}^n$ and family $\mathcal{S}$ of subsets of $[n]$

Goal: Given purported restrictions $\{g \restriction_S \mid S \in \mathcal{S}\}$ of $g \in \{0,1\}^n$, test if $g \in \mathcal{C}$

Agreement testing

Given local views $\{f_S \in \{0,1\}^S \mid S \in \mathcal{S}\}$
Choose random $A, B \in \mathcal{S}$
Accept if $f_A \restriction_{A \cap B} = f_B \restriction_{A \cap B}$; else reject

Want to show that if the test accepts whp, then some global $g \in \mathcal{C}$ satisfies $g \restriction_S = f_S$ for almost all $S \in \mathcal{S}$

Example: $\mathcal{S} = d$-dimensional subspace of $\mathbb{F}_2^n$ $\qquad (d \ll n)$

$A \cap B$ has dimension $d-1$ or $d/2$ [IJKW10, KMS18, ...]

$g \mapsto \{g \restriction_S \mid S \in \mathcal{S}\}$ sometimes known as "direct product code"

Derandomization

Want a sparse collection of pairs (A, B)

Can take e.g. simplicial complex Y and $S = Y(d), A \cap B = Y(k) \qquad (d > k)$

Double sampler [DK17]

Tripartite graph on $U = [n], V \subseteq \binom{[n]}{k}, W \subseteq \binom{[n]}{d}$
Edges across adjacent layers to represent inclusion relation
Bipartite expanders on adjacent layers

Dinur–Kaufman constructed one with size $O_d(n)$ and showed agreement test is sound

Expansion follows from high dimensional expander

Other coding applications

Locally testable codes: Dikstein–Dinur–Harsha–Ron-Zewi [DDH+20], Kaufman–Oppenheim [KO21]

List decodable codes: Dinur–Harsha–Kaufman–Navon–Ta-Shma [DHK+19], Alev–Jeronimo–Quintana–Srivastava–Tulsiani [AJQ+20]

Explicit sum-of-squares lower bounds

Previous lower bounds to constraint satisfaction problems in the sum-of-squares hierarchy are based on random instances (e.g. random 3-SAT) [Gri01, Sch08, ...]

1. strong expansion of factor graph: For every $\leqslant \varepsilon n$ clauses, some clause has a unique variable
2. union bound over random assignment

No explicit lower bound before [DFHT20]

[DFHT20] instead constructed explicit lower bounds

Use other expansion properties than eigenvalue gap!

Open problems

More codes from high dimensional expanders?

Markov chain to sample common bases of two matroids?